Technical Comments

Comments on "An Analytic Solution for Entry into Planetary Atmospheres"

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RECENTLY Citron and Meir[2] obtained an analytic solution for entry into planetary atmospheres. Following Eggers,[3] they transformed the equations of entry dynamics into

$$(d^2f)/(dz^2) = (\alpha \cos^2\gamma/f)[\exp(z - z_e) - 1] - \xi \cos\gamma \quad (1)$$

$$\sin\gamma = \zeta(df/dz) \quad (2)$$

where $f = \rho/\rho_0$, $z = -\ln(V^2/rg)$, $\alpha = (\beta B^2)/(rg^2\rho_0^2)$, $\zeta = (\rho_0 g)/(\beta B)$, $\xi = (\beta BC_L)/(2g\rho_0 C_D)$, and $B = (mg)/(C_D A)$. To obtain an analytic solution, Citron and Meir[2] assumed that $\cos\gamma = \cos\gamma_e$ in Eq. (1) and expanded f in a power series in terms of the variable z, i.e., they assumed that

$$f = f_e + f_e'z + A_2z^2 + A_3z^3 + \ldots \quad (3)$$

$$f' = f_e' + 2A_2z + 3A_3z^2 + \ldots \quad (4)$$

Neglecting terms of higher order than z^2, they solved Eq. (4) for A_2 and substituted its value in Eq. (3) to obtain

$$f' + f_e' = (2/z)(f - f_e) \quad (5)$$

Multiplying Eq. (5) by f'' and integrating leads to

$$\frac{f'^2}{2} + f_e'f' - \frac{3}{2}f_e'^2 = \int_0^z \frac{2}{z} ff''\left(1 - \frac{f_e}{f}\right) dz \quad (6)$$

They approximated the integral in Eq. (6) by neglecting "small terms." However, in the process of approximation, they neglected the term $(-ze^{-z_e})$ which comes from $\{-e^{-z_e} \times (f_e/f_e') \ln[1 + (f_e'/f_e)z]\}$ although they retained terms, which are of higher order than z, for example

$$e^{-z_e} \sum_{n=2}^{\infty} \frac{z^n}{nn!}$$

The importance of this neglected term depends on the value of f_e/f_e' compared to unity. In general, neglecting f_e/f_e' compared to unity is not reasonable because it can be very large. In physical variables,

$$f_e/f_e' = (\rho_e g)/(\beta B \sin\gamma_e) \quad (7)$$

Using, for illustration, the numerical values used by Citron and Meir[2] in obtaining Figs. 2 and 3, we find that f_e/f_e' is infinite because $\gamma_e = 0$.

Therefore, in general, the coefficient of z^2 in Eq. (49) of Citron and Meir[2] is incorrect, and, hence, their solution is correct only to order z rather than z^2 as they claimed. In that respect, the solution of Citron and Meir[2] has an order of error no less than the solution $(f = f_e + f_e'z)$ of Allen and Eggers[1] when f_e/f_e' is not very small.

Indeed, expanding the solution in power series of higher order than z will improve the solution of Allen and Eggers[1]: $f = f_e + f_e'z$. In order to obtain a solution correct to order z^3, we let $\cos\gamma$ in Eq. (1) be variable, substitute the series (3) in Eqs. (1) and (2), and equate the coefficients of equal powers of z to obtain

$$A_2 = (\{\alpha \cos^2\gamma_e[\exp(-z_e) - 1]\}/2f_e) - \tfrac{1}{2}\xi \cos\gamma_e \quad (8)$$

$$A_3 = \frac{\alpha \cos^2\gamma_e \exp(-z_e) - \xi f_e' \cos\gamma_e}{6f_e} + \left\{\frac{\xi\zeta^2 f_e}{\cos\gamma_e} - 2\alpha\zeta^2 [\exp(-z_e) - 1] - 1\right\}\left(\frac{f_e'A_2}{3f_e}\right) \quad (9)$$

The constants f_e and f_e' can be determined from the initial conditions. Once f is known, the altitude h can be determined from the isothermal condition $f = \exp(-\beta h)$, whereas γ can be determined from Eq. (2).

References

[1] Allen, H. J. and Eggers, A. J., Jr., "A study of the motion and aerodynamic heating of ballistic missiles entering the earth's atmosphere at high supersonic speed," NACA Rept. 1381 (1958).

[2] Citron, S. J. and Meir, T. C., "An analytic solution for entry into planetary atmospheres," AIAA J. 3, 470–475 (1965).

[3] Eggers, A. J., Jr., "The possibility of a safe landing," *Space Technology* (John Wiley & Sons, Inc., New York, 1959), Chap. 13

Received June 23, 1965; revision received September 27, 1965.
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Reply by Author to A. H. Nayfeh

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NAYFEH[1] makes two assertions in his comment. They are a) that a significant term of order $\bar{z}$ has been neglected in the approximate solution of Ref. 2, and hence the solution by Citron and Meier is no more accurate than that of Allen and Eggers $(f = f_e + f_e'\bar{z})$; and b) that expanding the solution in a power series, retaining higher order terms than $\bar{z}$, will usefully improve the solution of Allen and Eggers for small entry angles. In particular, Nayfeh develops the expansion through terms of order $\bar{z}^3$. Both of these assertions are incorrect.

Before proceeding to the analytical discussion, consider the example cited by Nayfeh to demonstrate his point. This is the case of tangential entry into the atmosphere at circular velocity, Figs. 2 and 3 of Ref. 2, where f_e/f_e' is infinite because $\gamma_e = 0$. The comparison given in Ref. 2† of the Citron-Meier solution for this case with the exact numerical solution is reproduced here in Figs. 1 and 2. In addition the results for the improved Allen and Eggers solution, developed in Ref. 1 by Nayfeh through terms of order $\bar{z}^3$, is also presented in Figs. 1 and 2.

Now, Nayfeh comments "the solution of Citron and Meir[2] has an order of error no less than the solution $(f = f_e + f_e'\bar{z})$ of Allen and Eggers[1] when f_e/f_e' is not very small." Yet, it is apparent from the results of Figs. 1 and 2 that the Citron-Meier solution is reasonably accurate, whereas the improved Allen and Eggers solution presented by Nayfeh is not. The reason for this failure of the series solution developed by Nayfeh is seen when the numerical values for the coefficients in

Received August 4, 1965; revision received December 1, 1965.
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† Equation (47) of Ref. 2 should read

$$f' = -\left(f_e' + \frac{J\bar{z}}{2}\right) + F(\bar{z})$$